Replicated Secret Sharing

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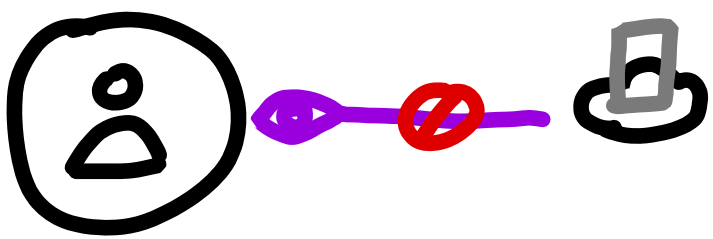
2025-09-15

Motivation

1. We have some “secret” values



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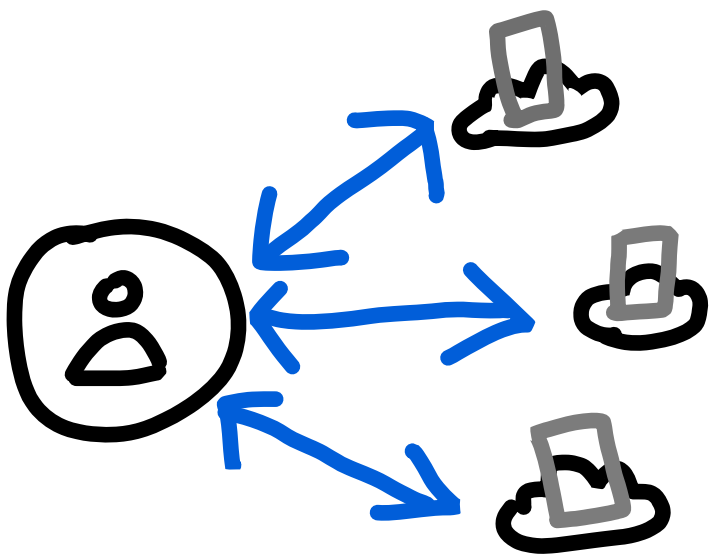

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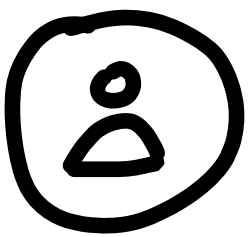

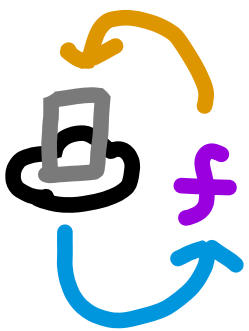

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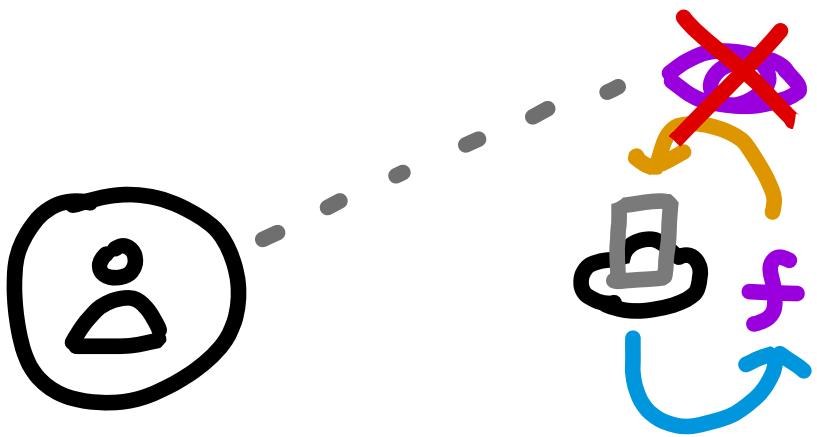
2. We want these values to be “replicated”
3. We have functions that should run remotely on secrets to produce new secrets
4. We don’t want any machine holding a “share” of the secret to learn the secret when doing computations



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For the ith participant, give it multiple shares $[s_i, \ldots, s_{i+t}]$:

- at most t participants can "fail" while leaving the secret recoverable
- at most $t - 1$ participants can lie while leaving the secret recoverable
- at least $n - t$ participants have to collude to learn the secret s

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If we choose $t = 1$,

Participant 1 receives $[9, -7]$

Participant 2 receives $[-7, 3]$

Participant 3 receives $[3, 9]$

Reconstruction

$$p_1 \leftarrow [9, -7] \qquad p_2 \leftarrow [-7, 3] \qquad p_3 \leftarrow [3, 9]$$

What happens if participants 1 and 2 collude?

- They see that they both have -7...

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What happens if participant 3 fails?

- The reconstruction of values from p_1 and p_2 is still $9 + (-7) + 3 = 5$, recovering the secret

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For example:

$$a = 5 = 9 + (-7) + 3 \qquad b = 8 = 2 + 1 + 5$$

We can calculate $c = a + b = 5 + 8$ as:

$$\begin{aligned} c &= (9 + 2) + (-7 + 1) + (3 + 5) \\ &= 11 + (-6) + 8 \\ &= 13 \end{aligned}$$

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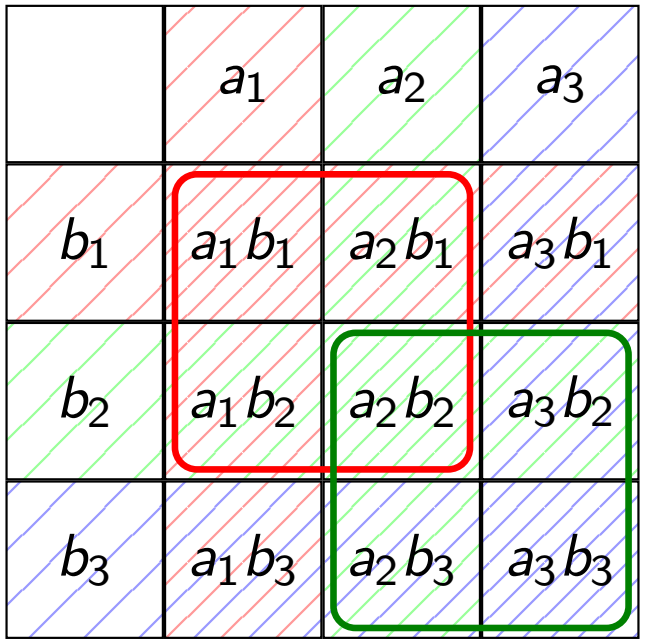
- We would like some definition of c_i s.t. each c_i is an n-sized sum of these products.

Multiplication on Secrets: $n = 3, t = 1$



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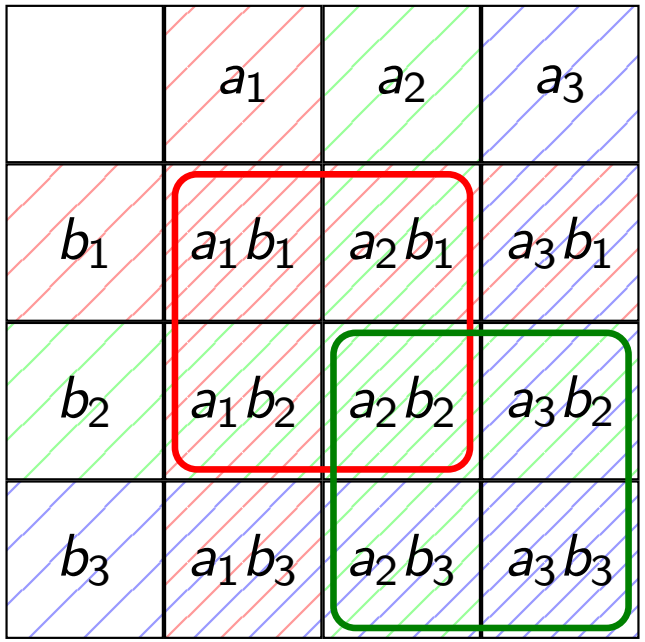

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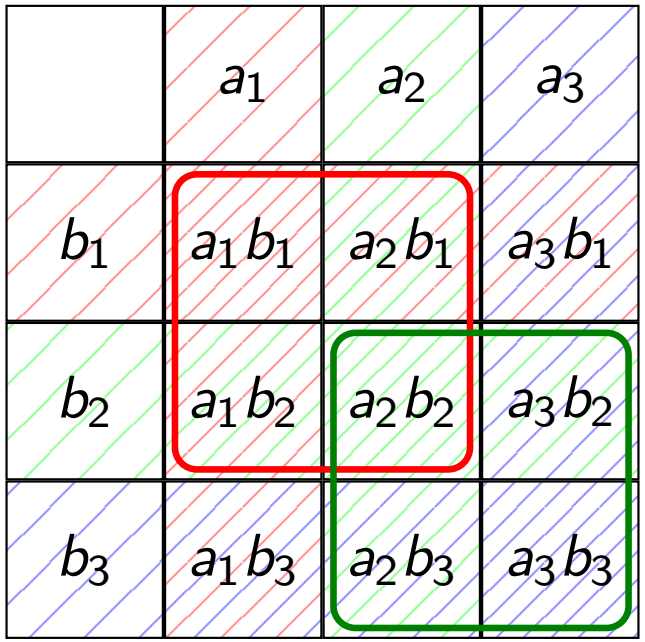

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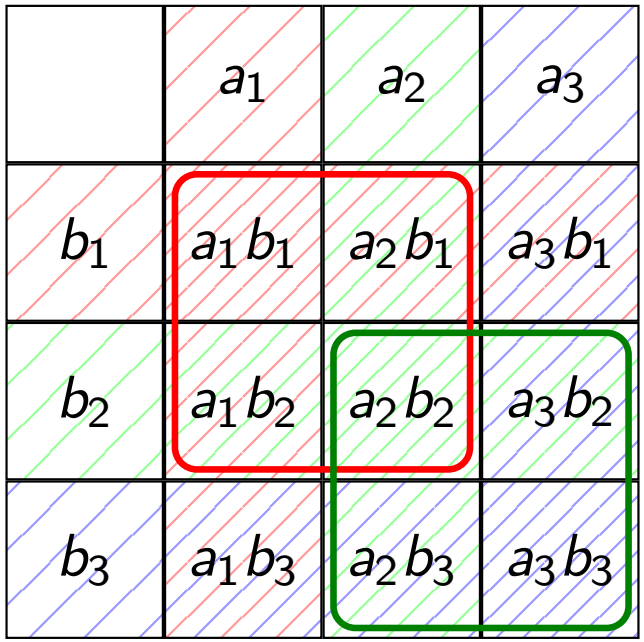

- Each participant can calculate 4 products
- Extra product is case $i = j$: resolve by giving $a_i b_i$ to participant i
- So $c_i = a_i b_i + a_i b_{i+1} + a_{i+1} b_i$

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One big issue:

Raw multiplication results aren't secure!

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2. Participant 3 knows:
   - a_1, a_3, b_1, b_3, and c_3, as it is participant 3
   - $c_1 = a_1 b_1 + a_1 b_2 + a_2 b_1$ as it received c_1 as part of multiplication

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3. This teaches participant 3 about $a_1 b_2 + a_2 b_1$
4. Which allows participant 3 to learn:
   - $(a_1 b_2 + a_2 b_1) \equiv a_2 b_1 \pmod{a_1} \Rightarrow a_2 b_1 b_1^{-1} \equiv a_2 \pmod{a_1}$
   - Likewise for $b_2 \pmod{b_1}$

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6. calculates $c_1' = c_1 + r_1 - r_3, c_2' = c_2 + r_2 - r_1$
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7. assumes the replicated share $[c_1', c_2']$ for the secret c

In the final sum, the random values cancel:

$$c' = c_1 + r_1 - r_3 + c_2 + r_2 - r_1 + c_3 + r_3 - r_2 = c$$

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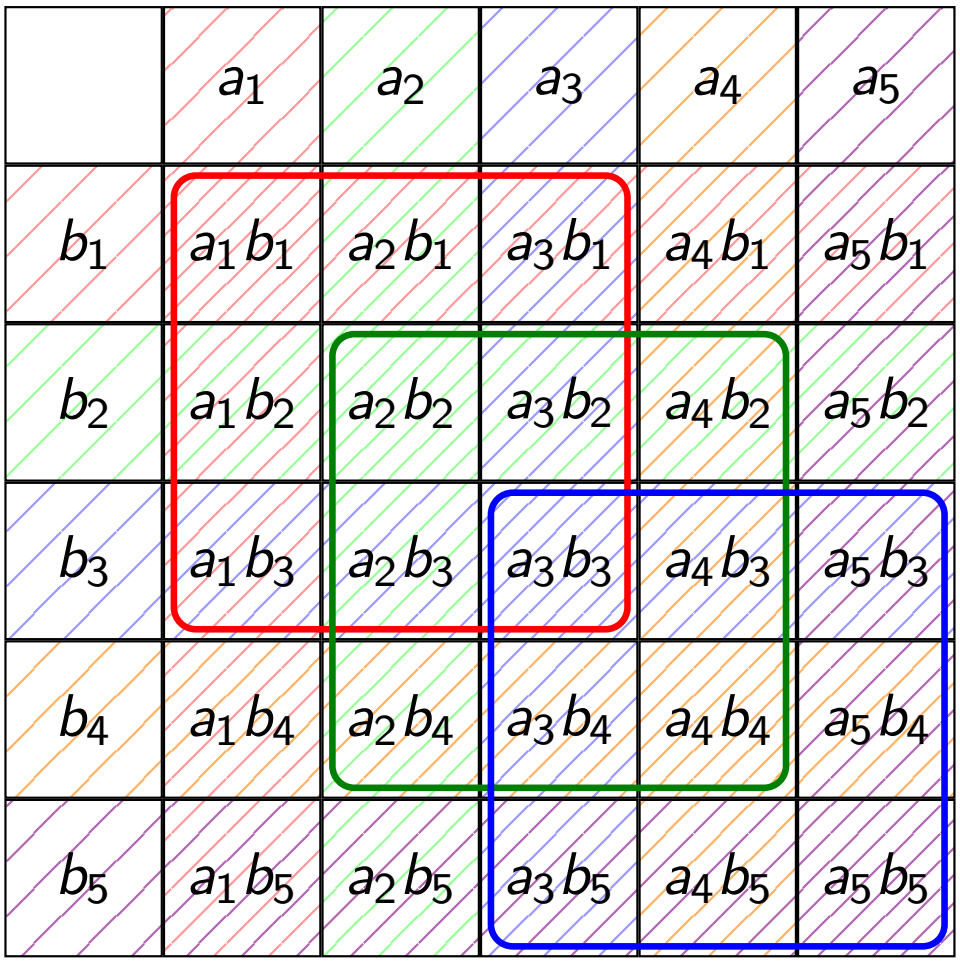
Needs one more honest participant than simple majority: $z < \frac{n-1}{2}$

Multiplication for Larger t



	a_1	a_2	a_3	a_4	a_5
b_1	$a_1 b_1$	$a_2 b_1$	$a_3 b_1$	$a_4 b_1$	$a_5 b_1$
b_2	$a_1 b_2$	$a_2 b_2$	$a_3 b_2$	$a_4 b_2$	$a_5 b_2$
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b_4	$a_1 b_4$	$a_2 b_4$	$a_3 b_4$	$a_4 b_4$	$a_5 b_4$
b_5	$a_1 b_5$	$a_2 b_5$	$a_3 b_5$	$a_4 b_5$	$a_5 b_5$

(Table for $n = 5, t = 2$.)

A participant's c_i is the sum of the products on the top-left border of their square. Each participant still generates just one random number, and sends $c_i + r_i$ to the t participants to its left and r_i to the t participants to its right.

The End :)